FROM A POISSON PROCESS MODEL TO A LAPLACIAN MARTINGALE PROCESS FOR EARTHQUAKE FORECASTING IN A REGION IN THE PHILIPPINES

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Abstract

The usual Poisson process for seismic hazard assessment is utilized as basis for constructing a Laplacian martingale process to respond to objections raised about the use of a Poisson model for forecasting earthquake occurrences. Seismic data from the Caraga Region of the Philippines were utilized in this study. The seismic signals themselves are shown to follow a power-law distribution with λ = 2.49915. Results show that the Laplacian martingale process predicted the arrival times of earthquakes with intensity 4 or greater with less than 1% relative error for the Caraga region. The unpredictability of the occurrence of earthquakes of such magnitudes appear to share the same unpredictability property with the occurrence of the nth prime in analytic number theoryto Ph.D. as well as aligned teaching assignments, research outputs and other scholarly works.

Keywords: *Poisson process, martingale process, inter-arrival times, counting process*

1.0 Introduction

The power law distribution:

$$f(x) = (\lambda - 1)\, x^{-\lambda}, x > 1, \lambda > 1 \qquad (1)$$

is used as a probability model for seismic signals in a given location (Richter and Guttenberg, 1956). Amitrano (2012) explains how the exponent changes with location on the earth's surface. Earthquakes with intensity 4 or greater are of interest since such tremors become physically perceptible to humans. Let E={├ x┤|x≥4}. Define

$$I(x_i) = \begin{cases} 1, if\ x_i \geq 4 \\ 0, else \end{cases} \qquad (2)$$

then:

$$P(E) = P(I(x_i) = 1) = P(x_i \geq 4\,) = 4^{1-\lambda} \qquad (3)$$

Let $x_1, x_2, \ldots, x_n$ be a sequence of seismic signals and let:

$$S_k = \sum_{i=1}^{k} I(x_i) \qquad k = 1, 2, \ldots, n \qquad (4)$$

Here, S_k counts the number of 1's up to time k and:

$$S_k \underset{\sim}{d}\ Binomial\ (k, p)$$

where *p=P(E)* is given in (3). More succinctly, the probability that k earthquakes of magnitude 4 or greater have occurred up to time n is:

$$P(S_n = k) = \binom{n}{k} \left(4^{1-\lambda}\right)^k \left(1 - 4^{1-\lambda}\right)^{n-k}, k = 0, 1, \ldots, n \qquad (5)$$

In this paper, we use a continuous – time formulation of (5) to construct two (2) stochastic processes, namely, the Poisson process and the Laplacian process to predict the occurrence of the (n+1)^th earthquake given the previous occurrences.

2.0 The Poisson Process Model

We introduce the Poisson process model in this section as used in the literature. There are many views about the use of a Poisson process in modeling earthquakes with some authors claiming that it is inappropriate because of its "memory-less" property, others have argued otherwise and provide evidence to support the assumption in seismic hazard assessment that earthquakes are Poisson processes [Reiter, 1990; Bozorgnia and Bertero, 2004;Lombardi et al., 2005; Kossobokov, 2006]. This assumption is routinely stated yet seldom tested or used as a constraint when fitting frequency-magnitude distributions. Moreover, the assumption of a Poisson process leads to a forecast that adds the same quantity every time to the previous observation. Wang et al. (2014) provided new evidence and perspective on the Poisson process model for earthquakes to over 55,000 earthquake events in Taiwan since 1900. Wang et. Al. (2014) showed that the Poissonian hypothesis applied to local magnitude (ML)≥3.0 earthquakes around Taiwan with a mean annual rate as high as 1,600 is clearly rejected, but as far as ML≥7.0 earthquakes with a mean rate of 0.35 per year are concerned, the same hypothesis is statistically accepted for modeling their temporal randomness.

Let $\{N(t): t \geq 0\}$ be a counting process defined by:

$$N(t) = \sum_{j=0}^{t} I(x_j) \qquad (6)$$

where:

a) *N(0)=0;*
b) *N(t)* has independent increments;
c) *N(t)* over any interval of length *t* is a Poisson random variable with parameter β *t*.

Here $\beta=4^{1-\lambda}$ and by independent increments we mean that for any two disjoint intervals J_k and J_l, $J_k \cap J_l = \emptyset$, the events over J_k are independent of the events in J_l. From (6),

$$P(N(t) = k) = \frac{e^{-\beta t}(\beta t)^k}{k!} \quad , k = 0,1,2, \ldots \qquad (7)$$

Greenhough (2007) stated that (7) can be thought of as an approximation to a binomial distribution (5) when n→∞ while np →constant, which essentially means that the probability of success p→0.

The function N(t) is a discontinuous step function. Let $T_1, T_2, \ldots, T_q$ be the jump discontinuities of N(t) and q=[βt], where [.] is the greatest integer function, $T_1 \leq T_2 \leq \ldots \leq T_q$ are the arrival times of the earthquake events E.

Theorem 1. Under the Poisson model (7), the inter-arrival times $t_k = T_k - T_{k-1}$, k=1,2,…, n, $T_0=0$, are exponentially distributed with parameter β.

Proof. Consider $t_1 = T_1$ and note that:

$P(T_1>t)$=P(first arrival happens after time t)
=P(no arrival between 0 and time t)
$=P(N(t)=0)=e^{-\beta t}$,

hence, $t_1 \overset{d}{\sim}$ Exp (β). Next, let $t_2 = T_2 - T_1$. Let s>0 and t>0 and the intervals (0, s] and (s,s+t] are independent.

$P(t_2>t|t_1=s)$=P(no arrival in (s,s+t]| t_1=s)
=P(no arrival in (s,s+t]) by independent increments
$=e^{-\beta t}$

and so:

$$F_{t_2}(t)=1-e^{-\beta t}.$$

The rest follows by the same argument.

Theorem 1 can be used to forecast the occurrence of the nth earthquakes since:

$$\text{pred}\,(T_n)=T_{n-1}+E(t_n), \qquad (8)$$

where T_{n-1} is the most recent occurrence and $t_n \overset{d}{\sim}$ Exp (β). However, this is precisely the objection raised by other seismological experts viz. adding the same quantity to the previous arrival time as a forecast function.

The parameter λ of the generating power law distribution needs to be estimated. Its maximum-likelihood estimator is:

$$\hat{\lambda} = 1 + \frac{n}{\sum_{i=1}^{n}(x_i)} \qquad (9)$$

Theorem 2. The maximum likelihood estimator of λ given in (9) converges in probability to λ.

Proof. By the Strong Law of Large Numbers (SLLN), we have:

$$\frac{1}{n}\sum_{i=1}^{n} log(x_i) \rightarrow E(\log(x)).$$

Now, by integration:

$$E(\log(x)) = \int_1^{\infty} \log(x)(\lambda - 1)\, x^{-\lambda}\, dx$$

$$= \frac{1}{\lambda-1}$$

Slutsky's theorem states that:

$$\hat{\lambda} = 1 + \frac{n}{\sum_{i=1}^{n} log(x_i)} = 1 + \frac{1}{\frac{1}{n}\sum_{i=1}^{n} log(x_i)} \rightarrow$$

$$1 + \frac{1}{E(\log(x))} = 1 + \frac{1}{\frac{1}{\lambda - 1}} = \lambda \; as \; n \rightarrow \infty \; ■$$

3.0 The Laplacian Process

The prediction given in (8) depends only on the latest occurrence (T_{n-1}) time of the earthquake event of interest. However, the occurrence of an earthquake of large intensity, say x≥6.0 is often followed by aftershocks of magnitudes greater than 4.0. This suggests that the prediction equation will have to involve occurrences of the event other than just the latest.

As before, let

$$t_n = T_n - T_{n-1} \qquad \text{for } n = 1, 2, \dots, N, T_0 = 0.$$

Define

$$d_k = t_{n+k} - t_n \qquad \text{for } k = 1, 2, \dots, N - n, \qquad (10)$$

as **occurrence depth at level k** or simply **level k depth occurrence**. For *k*=1, 2 we have:

$$d_1 = t_{n+1} - t_n = T_{n+1} - T_n - (T_n - T_{n-1}) = T_{n+1} - 2T_n + T_{n-1}$$

$$d_2 = t_{n+2} - t_n = T_{n+2} - T_{n+1} - (T_n - T_{n-1}) = T_{n+2} - T_{n+1} - T_n + T_{n-1} \qquad (11)$$

From (11), we obtain:

$$T_{n+2} = T_{n+1} + T_n - T_{n-1} + d_2 \qquad (12)$$

or

$$T_n = T_{n-1} + T_{n-2} - T_{n-3} + d$$

where d is random variable with the same distribution as d_2. Note that the sequence $\{d_{1,n}\}_{n=1}^{N}$ is a sequence of dependent random variables while the sequence $\{d_{2,n}\}_{n=1}^{N}$ is a sequence of independent random variables.

Theorem 3. The sequence $\{d_{2,n}\}$ are iid Laplace random variables with μ=0 and b=β:

$$f_{d_2}(x) = \frac{\beta}{2} e^{-\beta|x|} \qquad , -\infty < x < \infty$$

Proof.
We observe that d_2=Y-Z where Y and Z are iid exp (*β*). Hence d_2 is the difference of two (2) independent exponential random variables. The mgf of Y is:

$$m_y(t) = E(e^{ty}) = \frac{\beta}{\beta - t}$$

and so:

$$m_{d_2}(t) = \left(\frac{\beta}{\beta-t}\right)\left(\frac{\beta}{\beta+t}\right) = \frac{\beta^2}{\beta^2 - t^2}$$

which is the mgf of a Laplace random variable as can be easily verified.

We could, if we wanted to, use (12) to predict T_n. as follows:

$$pred(T_n) = T_{n-1} + T_{n-2} - T_{n-3} + E(d_2) \qquad (13)$$

However, since $E(d_2)=0$, (13) will under-estimate T_n. For this reason, we choose to put the forecasting problem in the context of a martingale.

Filtration and Martingale Processes

In order to make use of (13) in predicting T_{n+1}, we model the depth d_2 as a martingale process. This section develops this idea.

Definition 1. Let $X \neq \emptyset$. A sigma algebra (σ-algebra) $\mathcal{F}$ on X is a class of subsets of X such that:

(a) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
(b) If A, B $\in \mathcal{F}$, then $A \cup B \in \mathcal{F}$
(c) $X \in \mathcal{F}$.

Definition 2. A sequence $\{\mathcal{F}_n : n \in N\}$ of σ-algebras such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for every $n \in N$ is called a filtration.

For example, if we have a stochastic process $\{X_t\}$ and observed $X_1, X_2, \ldots, X_n$. Let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ be the smallest σ-algebra that makes the random variable $X_1, X_2, \ldots, X_n : \Omega \to R$ measurable. The σ-algebra $\mathcal{F}_n$ represents the amount of information available up to time n. Next, we connect the idea of a filtration and stochastic processes.

Definition 3. Let $\{X_n : n > 0\}$ be a filtration i.e. $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{A}$ of $\sigma-$ algebras. We call X_n an $\mathcal{F}_n$ −adapted process if X_n is $\mathcal{F}_n$ − measurable for all $n \geq 0$.

In practical terms, we consider a stochastic process indexed by discrete time T=N. For each *n,* X_n is a realization from some random process $\mathcal{F}_n$ e.g. Wiener process, Brownian motion.

Definition 4. A discrete process $\{X_n : n > 0\}$ is called a martingale process adapted to the filtration $\{\mathcal{F}_n : n > 0\}$ if:

(a) X_n is adapted to $\mathcal{F}_n$;
(b) $E(|X_n|) < \infty$ for all n, and
(c) $E(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}$ almost surely for all $n \geq 1$.

Theorem 4: The sequence $\{d_2^n\}$ adapted to the filtration $\{\mathcal{F}_n : n > 0\}$ characterized by a Laplace distribution with μ=0 and scale parameter $\beta = 4^{1-\lambda}$ is a martingale.

Proof.

We verify the properties of a martingale. We show the first property by induction on n:

For n = 1: $\mathcal{F}_1 = \sigma(x_1)$

For n = 2: $\mathcal{F}_2 = \mathcal{F}_1 \cup \sigma(x_2) \Rightarrow \mathcal{F}_1 \subset \mathcal{F}_2$

Assume that it also holds for n = k, that is,

$$\mathcal{F}_k \subset \mathcal{F}_{k+1} = \mathcal{F}_k \cup \sigma(x_{k+1})$$

Now, for n = k+1,

$$\mathcal{F}_{k+2} = \mathcal{F}_{k+1} \cup \sigma(x_{k+2}) \Rightarrow \mathcal{F}_{k+1} \subset \mathcal{F}_{k+2}$$

Therefore, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Hence, x_n is adapted to $\mathcal{F}_n$.

Note that as $n \to \infty, \mathcal{F}_n \to \mathcal{L}_n = \mathcal{A}$, the Laplace process.

Next, for large n, the filtration $\mathcal{F}_n \to \mathcal{L}_n = \mathcal{A}$

$$E(|x|) = \int_{-\infty}^{\infty} |x| e^{-|x|} dx < \infty$$
$$= 2\int_0^{\infty} x e^{-x} dx$$
$$= 2[-xe^{-x} - e^{-x}]_0^{\infty}$$
$$= 2(1)$$
$$= 2 < \infty$$

Finally, we show that $E(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}$ almost surely for all $n \geq 1$

$$E(x_n) = E(x_{n-1}) + E(x_n - x_{n-1})$$
$$E(x_n) = E(x_{n-1}) = x_{n-1} \text{ with respect to } \mathcal{F}_{n-1}$$

4.0 Simulation

We use Equation (13) to predict the (n+1)th occurence given the previous three(3) occurrences of earthquakes of magnitudes greater than or equal to 4.0. Using a Laplace filtration , we generated a random number δ from a Laplace distribution whose location parameter equals the immediate past occurence depth value. The predicted (n+1)th arrival time is:

$$pred(T_{n+1}) = T_n + T_{n-1} - T_{n-2} + \delta \quad (12)$$

For each T_{n+1} we repeated process (12) one hundred (100) times. The absolute error:

$$\text{Absolute error} = \left| \, pred(T_{n+1}) - actual(T_{n+1}) \, \right|$$

is computed and averaged out over 100 repetitions. The mean absolute relative error is computed as the average of :

$$\text{Absolute relative error} = \frac{Absolute\ Error}{Actual\ Arrival\ Time}$$

The seed data used in this study were obtained from the Department of Science and Technology website of PHILVOLCS from January, 2011 to April, 2013 per day for the Caraga Region (Surigao area). We imputed missing observations by assuming that for days between two(2) occurrences of the event E that had no information, seismic readings of less than 4.0 intensity are randomly imputed.

Figure 1 shows the histogram of the data set from PHILVOLCS:

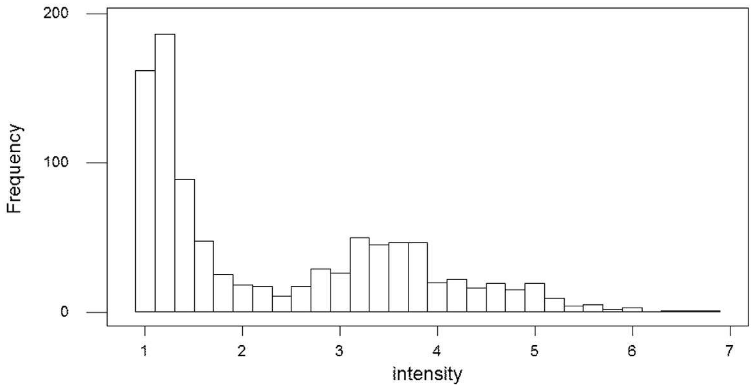

Figure 1: Histogram of the Seismic Readings in the Caraga Region

Figure 2, on the other hand, shows the plot of the summatory function N(t) for t=1,2,3,…,n.

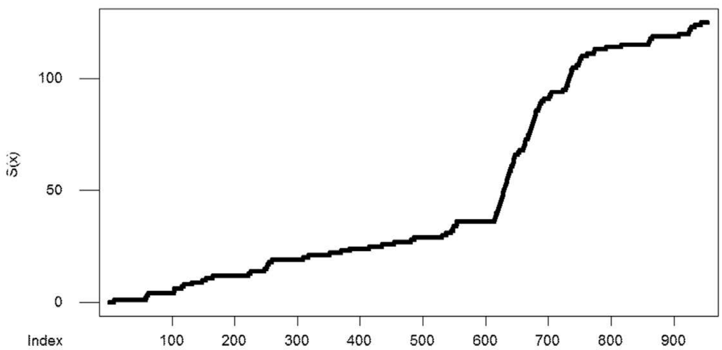

Figure 2: Plot of N(t) versus t

Figure 3 gives the arrival times of earthquakes of magnitude greater than or equal to 4 (from 1st arrival to 123rd arrival).

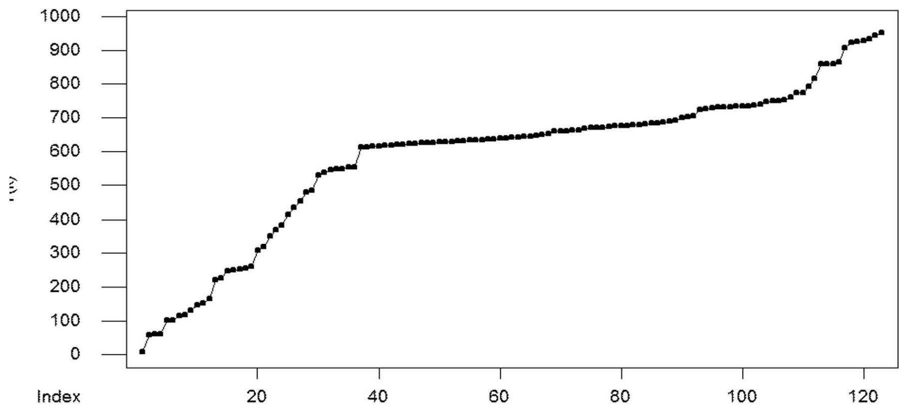

Figure 3: Arrival Times of Earthquakes of Magnitude 4 or greater

Figure 4 provides information on the distribution of the inter-arrival times of the earthquake events of interest:

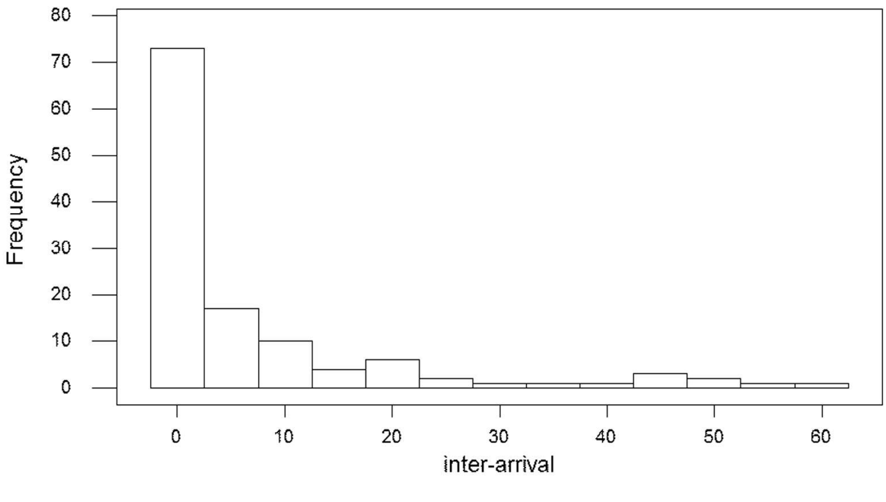

Figure 4: Histogram of the inter-arrival times

Figure 5 displays the histogram of the occurence depth d (2):

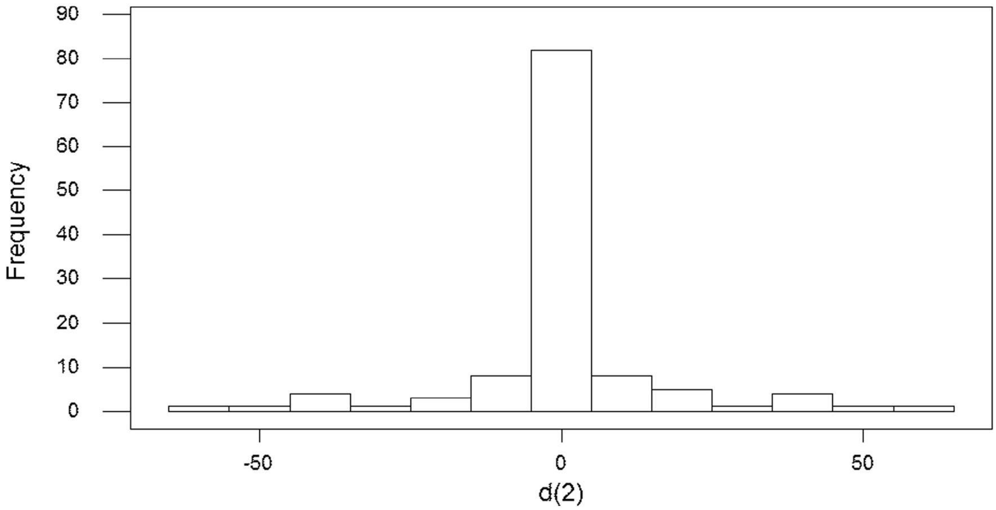

Figure 5: Histogram of the depth occurrence level 2

The histogram of the original observations as shown in Figure 1 is modeled as a power law distribution. The maximum likelihood estimator of λ is 2.49925 (or about 2.5). This gives the following value of β:

$$\beta = 4^{1-2.49925} = .12513$$

while the actual number of arrival times is 123.

Table 1. Actual versus Predicted Arrival Times Using the Exponential and Laplacian Models

N	nth arrival time	Exponential	Exponential Error	Laplacian Martingale	Laplace Error
120	928	933.992	5.992	928.089	0.089
121	934	935.992	1.992	932.186	1.8137
122	944	941.992	2.008	944.11	0.10985
123	954	951.992	2.008	954.037	0.03704

Simulation results show that the Laplacian martingale prediction model provides a better prediction of the occurrence of the nth earthquake with magnitude 4 or greater. Table 2 shows the average absolute error for the four(4) last occurrence times of earthquakes in the Caraga region:

Table 2: Mean Absolute Error for the Four(4) Occurrence Times Predictions

Variable	**N**	**Mean**	**StDev**
Exponential	4	3.000	1.995
Laplace Martingale	4	0.512	0.868

5.0 Conclusion

A Laplacian martingale process is a suitable model for predicting the occurrence of the nth earthquake of magnitude 4 or greater in the Caraga region of the Philippines. The usual Poisson process model for seismic signals appears to be inferior in terms of predicting the nth occurrence of an earthquake as compared to the martingale process proposed in this paper. The unpredictability of the occurrence of earthquakes of such magnitudes appear to share the same unpredictability property with the occurrence of the nth prime in analytic number theory (Frias and Padua, 2016) since in both instances, the Laplacian martingale process delivered very close prediction to the nth occurrence of an event.

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