ISOLATE DOMINATION OF SOME SPECIAL GRAPHS

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ABSTRACT

A dominating set S of G is called an isolate dominating set of G if the subgraph $\langle S\rangle$ induced by S has an isolated vertex. In this paper, we investigate the isolate domination of some special graphs such as cocktail party graph, crown graph, gear graph, and jump graph. Characterizations and exact values of isolate domination number were also derived for these graphs.

Keywords: *domination, isolate domination, cocktail party graph, crown graph, gear graph, jump graph*

1.0 Introduction

Throughout this paper, we only consider a *simple graph* (a graph without loop or multiple edges). For a graph G, we denote $V(G)$ and $E(G)$ as *vertex set* and *edge set* of G, respectively. Two vertices u and v are said to be *neighbor* or *adjacent* (denoted by uv) if $uv \in E(G)$. The set of all neighbors of u is denoted by $N(u)$ and it is called the *open neighborhood* of u, that is, $N(u) = \{x \in V(G): xu \in E(G)\}$. The *closed neighborhood* of u, denoted by $N[u]$, is given by $N[u] = N(u) \cup \{u\}$. An *induced subgraph* of G (simply, *subgraph* of G) is a graph with vertices less than or equal to $V(G)$ such that adjacencies and non-adjacencies of any two vertices in G are preserved. A subgraph of G induced by S is denoted by $\langle S\rangle$. The complement of a graph G, denoted by $\bar{G}$, is a graph of the same vertices as G such that any two vertices are adjacent if and only if they are not adjacent in G. Any undefined terms not specified in this paper, may see Harary (1969).

We say that a subset $S \subseteq V(G)$ is a dominating set of G if for all vertex $x \in S$, there exists a vertex $v \in V(G)\backslash S$ such that $uv \in E(G)$. The minimum cardinality of the dominating set of G is called a domination number of G, denoted by $\gamma(G)$ Haynes et al (1998). If a subgraph $\langle S\rangle$ induced by a dominating set S of G has an isolated vertex, then we say that S is an isolate dominating set of G. The minimum cardinality of an isolate dominating set is called isolate domination number, denoted by $\gamma_0(G)$. The concept of isolate domination was introduced by Hamid & Balamurugam (2013) and further studied by Arriola (2015).

Let us take some examples to illustrate the concept of domination and isolate domination. Consider the graph G as shown in Figure 1. Sets $S_1 = \{u_1\}$ and $S_2 = \{u_1, u_2\}$ are both dominating sets, and also both an isolate dominating sets.

Since the cardinality of S_1 is less than the cardinality of S_2, it follows that $\gamma_0(G) = 1$.

A *complete graph* is a graph on which every pair of vertices is connected by an edge. The complete graph with n vertices is denoted by K_n. Figure 2 shows a complete graph with 5 vertices. A dominating set or isolate dominating set of a certain graph need not be unique. For instance, the set of any single vertex of a complete graph constitute an isolate dominating set.

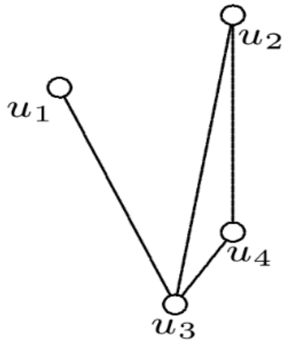

Figure 1 Simple Graph

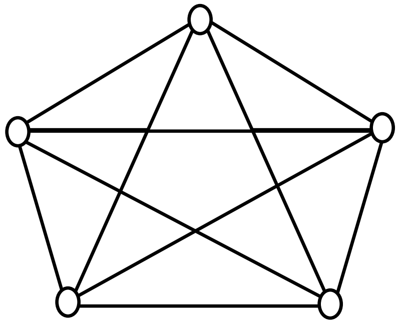
Figure 2: Complete Graph with 5 vertices

2.0 Main Results

In this section, we discuss some special graphs and illustrate these concepts through examples. Characterizations of the isolate domination and some exact values of its corresponding isolate domination number are also discussed in detailed.

2.1 Cocktail Party Graph

Biggs (1993) introduced the concept of cocktail party graph.

Definition 2.1.1

Let V^1 and V^2 be two sets of n vertices, respectively, such that every vertex in V^1 has a vertex pair in V^2. If $\langle V^1 \rangle \cong \langle V^2 \rangle \cong K_n$ such that each vertex in $\langle V^1 \rangle$ is connected by an edge to every vertex in $\langle V^2 \rangle$ except the paired ones, then we say that such graph is a *cocktail party graph*, denoted by $\overline{L_n}$.

Consider the graph shown in Figure 3. This graph illustrates a cocktail party graph since both subgraphs $\langle \{u_1, u_2, u_3, u_4\} \rangle$ and $\langle \{v_1, v_2, v_3, v_4\} \rangle$ are complete, and corresponding paired vertices are not connected by an edge. The following sets are the isolate dominating sets: $S_1 = \{u_1, v_1\}$, $S_2 = \{u_2, v_2\}$, $S_3 = \{u_3, v_3\}$, and $S_4 = \{u_4, v_4\}$.

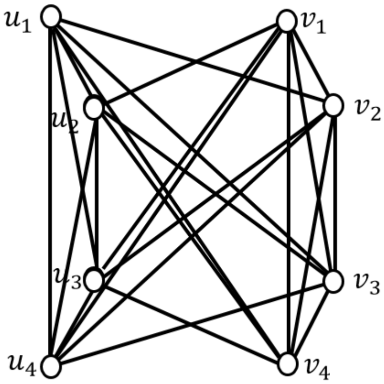

Figure 3: Cocktail Party Graph

Theorem 2.1.2

Any isolate dominating set of a cocktail party graph $\overline{L}_n$ of order $2n$ has size two. In particular, $\gamma_0(\overline{L_n}) = 2$.

Proof: Suppose S is an isolate dominating set of $\overline{L_n}$ such that $|S| \neq 2$. Suppose further that $|S| = 1$ and let $S = \{u\}$. All vertices are dominated by u except its vertex pair. Thus, S is not a dominating set of $\overline{L_n}$, a contradiction. Also, suppose $|S| \geq 3$. If $S \subseteq V^1$ (similarly, if $S \subseteq V^2$), then S has no isolated vertex, a contradiction. If $S \cap V^1 \neq \emptyset$ and $S \cap V^2 \neq \emptyset$, then S has no isolated vertex, a contradiction. Thus, $\gamma_0(\overline{L_n}) = 2$. ■

2.2 Crown Graph

The crown graph is similar to the cocktail party graph but differs only on its subgraphs. Both subgraphs V^1 and V^2 of a crown graph are complement of complete graphs. This crown graph was first studied by Brouwer et. al. (1989) and showed that crown graph is distance—transitive and isomorphic to the complement of the rook graph.

Definition 2.2.1

Let V^1 and V^2 be two sets of n vertices, respectively, such that every vertex in V^1 has a vertex pair in V^2. If $\langle V^1 \rangle \cong \langle V^2 \rangle \cong \overline{K_n}$ such that each vertex in $\langle V^1 \rangle$ is connected by an edge to every vertex in $\langle V^2 \rangle$ except the paired ones, then we say that such graph is a *crown graph*, denoted by S_n^0.

Figure 4 is an example of a crown graph. The corresponding isolate domination number is also 2 and the isolate dominating set for this graph is characterize in the next theorem. The following remark is consequent to the given definition of a crown graph.

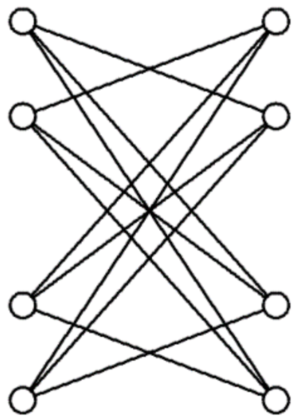

Figure 4: Crown Graph

Remark 2.2.2

The isolate dominating set of any crown graph has at least two elements.

Theorem 2.2.3

Let S_n^0 be a crown graph. Then a set $S \subseteq V(S_n^0)$ is an isolate dominating set of S_n^0 if and only if any of the following conditions hold:

i. $S \subseteq V^1 \cup \{x\}$, where $x \in V^2$ and its paired vertex in V^1 is in S such that $S = V^1$ whenever $x \notin S$;

ii. $S \subseteq \{y\} \cup V^2$, where $y \in V^1$ and its paired vertex in V^2 is in S such that $S = V^2$ whenever $y \notin S$.

Proof: Suppose S is an isolate dominating set of S_n^0. Let V^1 and V^2 be the vertex partition of S_n^0 as specified in Definition 2.2.1. If $|S| = 2$, the proof is straightforward. So, we may assume that $|S| \geq 3$ and consider the following cases:

Case 1: $S \cap V^1 = \emptyset$ (similarly, $S \cap V^2 = \emptyset$)

Then, $S \subseteq V^2$. If $S \neq V^2$, then there exists at least one element not in S. This implies that this particular element is not dominated by any element of S since V^2 is an empty graph. Hence, $S = V^2$.

Case 2: $S \cap V^1 \neq \emptyset$ and $S \cap V^2 \neq \emptyset$

If $|S \cap V^1| \geq 2$ and $|S \cap V^2| \geq 2$, then $\langle S \rangle$ is a connected graph which is a contradiction to our assumption that S is an isolate dominating set. Thus, either $|S \cap V^1| = 1$ or $|S \cap V^2| = 1$. This follows that $S \subseteq V^1 \cup \{x\}$, where $x \in V^2$ or $S \subseteq \{y\} \cup V^2$.

For the converse, assume that the given conditions hold. If $S = V^1$, then clearly S is an isolate dominating set. Also, if $S = V^2$, then S is an isolate dominating set. Suppose $S \subseteq V^1 \cup \{x\}$, where $x \in V^2$ and a paired vertex $x' \in V^1$ of x is in S. Then vertex x dominate any vertices in V^1 while x' dominate any vertices in V^2 and x' is an isolated vertex in $\langle S \rangle$. Similarly, for $S \subseteq \{y\} \cup V^2$, where $y \in V^1$ and a paired vertex $y' \in V^2$ of y is in S. ∎

Corollary 2.2.4

If S_n^0 is a crown graph of order $2n$, then $\gamma_0(S_n^0) = 2$.

Proof: Follows from Theorem 2.2.3. ∎

2.3 Gear Graph

Kirlangic (2009) derived some results about the rupture degree of gear graphs and further considered using the neighbor rupture degree by Bacak-Turan & Demirtekin (2017).

Definition 2.3.1

A *gear graph*, denoted by G_n is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph W_n (Brandstadt et. al, 1999).

For a gear graph G_n, we denote x_c as the central vertex, $V_w = N_{G_n}(x_c)$ be the set of all neighbors of x_c, and $\overline{V_w}$ be the set of all vertices not adjacent to x_c.

As an example, the first graph shown in Figure 4 is an example of wheel graph while the second graph is an example of gear graph.

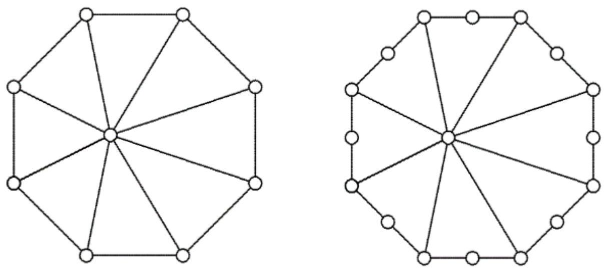

Figure 5: Wheel Graph and Gear Graph

Theorem 2.3.2

Let G_n be a gear graph of order $2n + 1$ with $n \geq 2$. Then the set $S \subseteq V(G_n)$ is an isolate dominating set of G_n if and only if either of the following holds:

i. $S \subseteq S_1 \cup S_2 \cup \{x_c\}$ such that $\overline{V_w} \subseteq N_{G_n}(S_1) \cup S_2$ and there exists $u \in S_2$ with $N_{G_n}(u) \cap S_1 = \emptyset$; or

ii. $S = S_1 \cup S_2$ such that $S_1 \neq \emptyset$, $N_{G_n}(S_1) \cup N_{G_n}(S_2) = V(G_n)$, and either there exists $u \in S_1$ with $N_{G_n}(u) \cap S_2 = \emptyset$ or there exists $v \in S_2$ with $N_{G_n}(u) \cap S_1 = \emptyset$,

where $S_1 \subseteq V_w$ and $S_2 \subseteq \overline{V_w}$.

Proof: Suppose S is an isolate dominating set of G_n. Let x_c be the central vertex of G_n. Consider the following cases:

Case 1: $x_c \in S$

Since S is a dominating set, every element of $\bar{V}_w$ has neighbor in S or it is in S. Take S_2 be the set of some elements of $\bar{V}_w$ that are in S and S_1 be the set of some elements of V_w that are neighbors of S_2 that are in S. This implies that $N_{G_n}(S_1) \cup S_2$ contain all elements of $\bar{V}_w$, that is, $\bar{V}_w \subseteq N_{G_n}(S_1) \cup S_2$. Since S is an isolate dominating set, there exists at least one vertex in $\bar{V}_w \cap S = S_2$ such that all its neighbor are not in S. Thus, if $u \in S_2$, then $N_{G_n}(u) \cap S_1 = \emptyset$.

Case 2: $x_c \notin S$

Since S is a dominating set, then at least one element of V_w must be in S. Take $S_1 = V_w \cap S$ and then $S_1 \neq \emptyset$. Moreover, since S is a dominating set, $N_{G_n}(S) = V(G_n)$. Take $S_2 = S \backslash S_1$. Then $V(G_n) = N_{G_n}(S_1) \cup N_{G_n}(S_2)$. Also, since S is an isolate dominating set, there exists an element in S, say x, such that $N_{G_n}(x) \cap S = \emptyset$. If $x \in S_1$, then $N_{G_n}(x) \cap S_2 = \emptyset$. If $x \in S_2$, then $N_{G_n}(x) \cap S_1 = \emptyset$.

The proof of the converse is straightforward. ■

Corollary 2.3.3

If G_n is a gear graph of order $2n + 1$, then $\gamma_0(G_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Proof: Since the gear graph without the central vertex is isomorphic to a cycle. We can choose first the vertex adjacent to the central vertex and then add every other three vertices to the set. Using Theorem 2.3.2, this set constitute an isolate dominating set for the gear graph. Formal argument is straightforward. ■

2.4 Jump Graph

Given any graph, the *line graph* is simply changing its edges and vertices into vertices and edges, respectively, such that adjacency is preserved. The *jump graph* may directly derived from the given graph or through the complement of its corresponding line graph. Maralabhavi et al (2013) particularly studied the graph theoretic properties of the domination number of jump graph.

Definition 2.4.1

The *jump graph* $J(G)$ of a graph G is the graph define on $E(G)$ and in which two vertices are adjacent if and only if they are not adjacent in G.

For example, given a graph G (see the first graph) in Figure 5. The corresponding jump graph of G is shown in second graph of Figure 5. Observe that all edges of G become vertices of $J(G)$ and its two vertices in $J(G)$ are adjacent if they are not edge adjacent in G. Jump graph is isomorphic to the complement of a line graph.

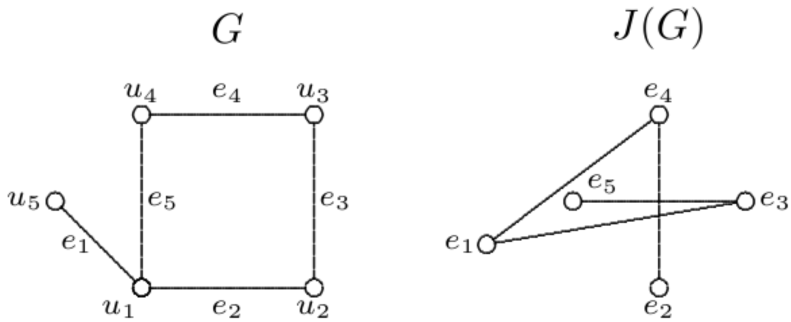

Figure 6: Graph G and its jump graph J(G)

Theorem 2.4.2

Let G be any graph of size $m \geq 1$. Then $\gamma_0(J(G)) = 1$ if and only if G contains a component P_2.

Proof: Suppos $\gamma_0(J(G)) = 1$ and let $S' = \{e\}$ be the isolate dominating set of $J(G)$. This implies that a vertex e dominates all other vertices in $J(G)$. Since e corresponds to an edge in G and has no adjacent edges, e is an isolated edge in G. Thus, G contains a component P_2. The converse is obvious. ∎

Theorem 2.4.3

Let G be a graph of size $m \geq 3$. Then $\gamma_0(J(G)) = 2$ if and only if there exists $u \in V(G)$ such that $deg_G(u) = 2$.

Proof: Suppose $\gamma_0(J(G)) = 2$ and let $S' = \{e_1, e_2\}$. Then e_1 and e_2 are adjacent edges such that either e_1 or e_2 is not adjacent to any edges in G. This implies that there is no other edge adjacent to both e_1 and e_2. Thus, if u is the common vertex connecting e_1 and e_2, then $deg_G(u) = 2$. The converse is straightforward. ∎

Corollary 2.4.4

Let $G \in \{P_n, C_n, F_n\}$ for $n \geq 3$. Then $\gamma_0(J(G)) = 2$.

Proof: Follows from Theorem 2.4.3.

Theorem 2.4.5

Let G be a graph of size $m \geq 2$. Then $\gamma_0(J(G)) = m$ if and only if G is a complete graph.

Proof: Arriola (2015) showed that for any graph G of order n, $\gamma_0(G) = n$ if and only if G is a null graph. Since the jump graph of a complete graph is a null graph, it follows that the isolate domination number of a jump graph of G is equal to its size. ■

3. Conclusion

In summary, this paper was able to derive and characterized the isolate dominating set of some special graphs such as cocktail party graph, crown graph, gear graph, and jump graph. It was further shown that the following are the exact value of the corresponding isolate domination number:

1. The isolate domination number of any cocktail party graph is 2.
2. The isolate domination number of any crown graph is 2.
3. The isolate domination number of a gear graph with n vertices is $\left\lceil \frac{2n}{3} \right\rceil$.

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